Discrete series for symmetric spaces over p-adic fields (Automorphic forms, trace formulas and zeta functions)

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数理解析研究所講究録
第 1767 巻 2011 年 14-24

Discrete series for symmetric spaces over p-adic fields

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0 Introduction

Let $\mathbf{G}$ be a connected reductive group over a non-archimedean local field F equipped with an F-involution $\sigma : \mathbf{G} \to \mathbf{G}$, and $\mathbf{H}$ the subgroup of all σ-fixed points of $\mathbf{G}$. The quotient space G/H of F-points $G = \mathbf{G}(F)$ by $H = \mathbf{H}(F)$ is called a symmetric space over F. We are interested in representations of G which can be realized in the space of functions on G/H. Such representations are said to be *H-distinguished*. We are concerned especially with representations which can be realized in the space $L^2(G/Z_G H)$ (in rough notation) of square integrable functions on $G/Z_G H$, where Z_G denotes the center of G. An irreducible representation having such a realization is said to be in *the discrete series for G/H*.

We give a criterion for such realizability, i.e., square integrability on G/H, in terms of *exponents* of Jacquet modules. The result has already appeared in [KT2]. In this report we give a brief survey of the main result of [KT2] (and also a part of our preceding work [KT1]). It is a symmetric space analogue of Casselman's well-known criterion for the group case ([C], which we recall below in Section 1). In our symmetric space analogue, we only consider Jacquet modules along *σ-split parabolics*, and consider exponents on *(σ, F)-split components* (see Section 3). The set of *relative exponents* used in our criterion is given in 5.1, and the main theorem is stated in 5.2. Several examples are included in the final section.

1 Casselman's criterion for the usual square integrability

Let F be a non-archimedean local field with the valuation ring $\mathcal{O}$ and the absolute value $|\cdot|_F$. Let $\mathbf{G}$ be a connected reductive group defined over F and $\mathbf{Z}$ the F-split component of $\mathbf{G}$, that is, the largest F-split torus in the center of $\mathbf{G}$. Let us write $G = \mathbf{G}(F)$, $Z = \mathbf{Z}(F)$.

Let (π, V) be a smooth representation of G. Suppose that $\pi|_Z$ is a unitary character of Z. Let $(\widetilde{\pi}, \widetilde{V})$ denote the contragredient of (π, V). The usual matrix coefficients of π are functions on G of the form

$$c_{v,\widetilde{v}}(g) = \langle \widetilde{v}, \pi(g^{-1})v \rangle \quad (g \in G)$$

for $v \in V$ and $\widetilde{v} \in \widetilde{V}$. The representation π is said to be square integrable (in the usual sense) if

$$\int_{G/Z} |c_{v,\widetilde{v}}(g)|^2 \, dg < \infty$$

for all $v \in V$ and $\widetilde{v} \in \widetilde{V}$.

We briefly say that P is a parabolic subgroup of G if P is the group of F-points of a parabolic F-subgroup $\mathbf{P} = \mathbf{P}$ of $\mathbf{G}$ etc, by abuse of terminology.

For a smooth representation (π, V) of G and a parabolic subgroup $P = MU$, let (π_P, V_P) denote the normalized Jacquet module of (π, V) along P: The space V_P is the quotient of V by the M-stable subspace

$$V(U) = \langle \{ \pi(u)v - v \mid u \in U, v \in V \} \rangle_{\mathbb{C}}.$$

Let $j_P : V \twoheadrightarrow V/V(U) = V_P$ be the canonical projection. Then M acts on V_P by

$$\pi_P(m) j_P(v) = \delta_P^{-1/2}(m) j_P(\pi(m)v)$$

where δ_P is the modulus of P. It is known ([C, §3]) that if π is finitely generated (resp. admissible), then so is the M-module π_P.

Let $\mathcal{X}(A)$ be the set of all quasi-characters of the F-split component A of the Levi subgroup M. For a $\chi \in \mathcal{X}(A)$, consider the *generalized χ-eigenspace*

$$(V_P)_{\chi,\infty} = \left\{ \overline{v} \in V_P \;\middle|\; \begin{array}{c} \text{There exists a } d \in \mathbb{N} \text{ such that} \\ \left(\pi_P(a) - \chi(a)\right)^d \cdot \overline{v} = 0 \text{ for all } a \in A \end{array} \right\}.$$

A quasi-character $\chi \in \mathcal{X}(A)$ is called *an exponent of* π_P if $(V_P)_{\chi,\infty} \neq \{0\}$. The set of all exponents of π_P is denoted by $\mathcal{E}_A(\pi_P)$.

The set $\mathcal{E}_A(\pi_P)$ is finite if π is finitely generated and admissible. One has a direct sum decomposition

$$V_P = \bigoplus_{\chi \in \mathcal{E}_A(\pi_P)} (V_P)_{\chi,\infty}.$$

Set

$$A^- = \{a \in A \mid |a^\alpha|_F \leqq 1 \text{ for all simple roots } \alpha\}.$$

Let us take up the following condition imposed on P:

$(\flat_P)$ $\qquad |\chi(a)| < 1$ for all $\chi \in \mathcal{E}_A(\pi_P)$ and all $a \in A^- \setminus Z\mathbf{A}(\mathcal{O}_F)$.

Now the well-known Casselman's criterion is stated as follows:

Theorem 1.1 (Casselman [C, 4.4.6]) *A finitely generated admissible representation π of G is square integrable if and only if the condition $(\flat_P)$ is satisfied for every parabolic subgroup P of G.*

2 H-square integrable representations

From now on we assume that the residual characteristic of F is not equal to 2. Let $\sigma : \mathbf{G} \to \mathbf{G}$ be an F-involution on $\mathbf{G}$. We put

$$\mathbf{H} = \{h \in \mathbf{G} \mid \sigma(h) = h\}, \quad \mathbf{Z}_0 = \left(\{z \in \mathbf{Z} \mid \sigma(z) = z^{-1}\}\right)^0,$$

We call $\mathbf{Z}_0$ *the (σ, F)-split component of* $\mathbf{G}$. We write $G = \mathbf{G}(F)$, $H = \mathbf{H}(F)$ etc.

A smooth representation (π, V) of G is said to be *H-distinguished* if the space $\mathrm{Hom}_H(\pi, \mathbb{C}) = (V^*)^H$ of H-invariant linear forms on V is nonzero. For a while, suppose that $\pi|_{Z_0}$ is a unitary character of Z_0, say, $\omega_0 : Z_0 \to \mathbb{C}_1^\times$. Let (π, V) be H-distinguished and take a non-zero H-invariant linear forms $\lambda \in (V^*)^H$. We consider functions $\varphi_{\lambda,v}$ on G/H for $v \in V$ defined by

$$\varphi_{\lambda,v}(g) = \langle \lambda, \pi(g^{-1})v \rangle \quad (g \in G).$$

Such functions are called *(H, λ)-matrix coefficients of* π. Note that these are not the matrix coefficients in the usual sense, but are *generalized* matrix coefficients, since

H-invariant linear forms are not smooth in general. We have an obvious equivariance for (H,λ)-matrix coefficients:

$$\varphi_{\lambda,v}(z_0gh) = \omega_0(z_0)^{-1}\varphi_{\lambda,v}(g) \quad \forall z_0 \in Z_0,\ g \in G,\ h \in H.$$

We also have $\varphi_{\lambda,\pi(g)v} = L(g)\varphi_{\lambda,v}$ (where $L(\cdot)\varphi$ denotes the left translation). Thus, for a fixed $\lambda \in (V^*)^H$, the set of functions $\{\varphi_{\lambda,v} \mid v \in V\}$ gives a realization of π in the space of functions on G/H.

Since $\mathbf{H}$ is reductive, H is unimodular. So the quotient space G/Z_0H carries a unique (up to constant) left G-invariant measure, denoted by $\int_{G/Z_0H} \dots dg$.

Definition 2.1 We say that π is (H,λ)-*square integrable* if $|\varphi_{\lambda,v}(\cdot)|$ is square integrable on G/Z_0H for all $v \in V$, namely, if

$$\int_{G/Z_0H} |\varphi_{\lambda,v}(g)|^2 dg < \infty$$

for all $v \in V$.

Remark **2.2** In our preceding work [KT1], we have defined that (π, V) is (H,λ)-*relatively cuspidal* if the support of $\varphi_{\lambda,v}$ is compact modulo Z_0H for all $v \in V$. So, by definition, (H,λ)-relatively cuspidal representations are (H,λ)-square integrable provided that ω_0 is unitary.

3 Tori and parabolics associated to σ

We recollect some notation and terminology for tori and parabolic subgroups associated to the involution σ. Basic reference is [HH].

Definition 3.1 (i) A parabolic F-subgroup $\mathbf{P}$ of $\mathbf{G}$ is said to be σ-split if $\mathbf{P}$ and $\sigma(\mathbf{P})$ are opposite, i.e., if $\mathbf{P} \cap \sigma(\mathbf{P})$ is a (σ-stable) Levi subgroup of $\mathbf{P}$.

(ii) An F-split torus $\mathbf{S}$ is said to be (σ, F)-split if $\sigma(s) = s^{-1}$ hold for all $s \in \mathbf{S}$.

Let us fix a maximal (σ, F)-split torus $\mathbf{S}_0$ of $\mathbf{G}$ and take a maximal F-split torus $\mathbf{A}_\emptyset$ containing $\mathbf{S}_0$. Then $\mathbf{A}_\emptyset$ is necessarily σ-stable, so σ acts naturally on $X^*(\mathbf{A}_\emptyset)$. Let $\Phi \subset \mathrm{X}^*(\mathbf{A}_\emptyset)$ be the root system of $(\mathbf{G}, \mathbf{A}_\emptyset)$. It is σ-stable. We can choose a

σ-basis Δ of Φ that has the property

$$\alpha > 0,\ \sigma(\alpha) \neq \alpha \implies \sigma(\alpha) < 0$$

in the corresponding order. The subset of all σ-fixed roots in Δ is denoted by Δ_σ .

Let $\mathbf{P}_\emptyset$ be the minimal parabolic subgroup corresponding to the choice of Δ. Standard parabolic subgroups $\mathbf{P}_I = \mathbf{M}_I\mathbf{U}_I$ (i.e., those containing $\mathbf{P}_\emptyset$) correspond to subsets I of Δ as usual. We can decide exactly when $\mathbf{P}_I$ is σ-split.

Lemma 3.2 ([HH, 2.6], [KT1, 2.5]) (i) *$\mathbf{P}_I$ is σ-split if and only if $I \supset \Delta_\sigma$ and the subsystem Φ_I of Φ generated by I is σ-stable. (In such a case we call I a σ-split subset.)*

(ii) *Any σ-split parabolic subgroup of $\mathbf{G}$ is written in the form $\gamma^{-1}\mathbf{P}_I\gamma$ for some σ-split subset $I \subset \Delta$ and $\gamma \in (\mathbf{M}_0\mathbf{H})(F)$, where $\mathbf{M}_0 = Z_{\mathbf{G}}(\mathbf{S}_0)$ denotes the centralizer of $\mathbf{S}_0$ in $\mathbf{G}$.*

Therefore, a minimal σ-split parabolic subgroup $\mathbf{P}_0$ of $\mathbf{G}$ can be given as the one corresponding to the minimal σ-split subset $I = \Delta_\sigma$. Alternatively it is given by $\mathbf{P}_0 = \mathbf{P}_\emptyset\mathbf{M}_0$. Note also that $\mathbf{M}_0$ is the σ-stable Levi subgroup of $\mathbf{P}_0$.

For a subset $I \subset \Delta$, let $\mathbf{A}_I$ be the F-split component of $\mathbf{M}_I$. If I is a σ-split subset, let $\mathbf{S_I}$ denote the σ-split part of $\mathbf{A}_I$, i.e., the identity component of $\mathbf{A}_I \cap \mathbf{S}_0$. We call $\mathbf{S}_I$ *the (σ, F)-split component of* $\mathbf{P}_I$. For a positive real number $\varepsilon \leqq 1$, we put

$$S_I^-(\varepsilon) = \left\{ s \in S_I = \mathbf{S}_I(F) \mid |s^\alpha|_F \leqq \varepsilon \ (\alpha \in \Delta \setminus I) \right\}$$

and

$${}_IS_0^-(\varepsilon) = \left\{ s \in S_0 = \mathbf{S}_0(F) \,\middle|\, \begin{array}{l} |s^\alpha|_F \leqq \varepsilon \ (\alpha \in \Delta \setminus I), \\ \varepsilon < |s^\alpha|_F \leqq 1 \ (\alpha \in I) \end{array} \right\}.$$

We abbriviate $S_I^- = S_I^-(1)$ and $S_0^- = S_{\Delta_\sigma}^-(1)$. We note that if $\alpha \in \Delta_\sigma$ and $s \in S_0$, then $s^\alpha = s^{\sigma(\alpha)} = \left(s^{-1}\right)^\alpha$, so that $|s^\alpha|_F = 1$.

Lemma 3.3 ([KT2, Lemma 1.6]) *For any $\varepsilon < 1$, one has*

$$S_0^- = \bigcup_{I \subset \Delta:\sigma\text{-split}} {}_IS_0^-(\varepsilon) \ (\textit{disjoint}).$$

It will turn out that the behaviors of H-matrix coefficients are determined essentially on S_0^-, and furthermore, on the subset ${}_IS_0^-(\varepsilon)$, they are connected to $M_I \cap H$-matrix coefficients of the Jacquet module along P_I.

4 Asymptotic behaviors of H-matrix coefficients

Let (π, V) be an admissible representation of G. Only when $P = MU$ is a σ-split parabolic subgroup, we have defined in [KT1] a linear mapping

$$r_P : (V^*)^H \to (V_P^*)^{M \cap H}$$

between the spaces of invariant linear forms. If $v \in V$ is a canonical lifting ([C, §4]) of $\bar{v} \in V_P$ with respect to a suitable σ-stable open compact subgroup, then $r_P(\lambda)$ for $\lambda \in (V^*)^H$ is well-defined by the relation

$$\langle r_P(\lambda), \bar{v} \rangle = \langle \lambda, v \rangle$$

(see [KT1, 5.3(2)]). The same mapping was constructed independently by N. Lagier [L] in a different manner. P. Delorme extended the construction of such mappings to any smooth representations by using Bernstein's *second adjointness theorem* in [D].

Now, through the mapping $r_P : (V^*)^H \to (V_P^*)^{M \cap H}$, the H-matrix coefficients of π are related to the $M \cap H$-matrix coefficients of the Jacquet module π_P as follows:

Proposition 4.1 ([KT2, 3.3]) *Let I be a σ-split subset of Δ and $P = P_I$ the corresponding σ-split parabolic subgroup with the (σ, F)-split component $S = S_I$. Let (π, V) be an H-distinguished admissible representation of G and $v \in V$, $\lambda \in (V^*)^H$. There exists a positive real number $\varepsilon \leqq 1$ such that*

$$\langle \lambda, \pi(s) v \rangle = \delta_P^{1/2}(s) \langle r_P(\lambda), \pi_P(s) j_P(v) \rangle$$

for all $s \in {}_I S_0^-(\varepsilon)$.

Remark **4.2** In [KT1, 6.2], we have shown the following criterion for (H, λ)-relative cuspidality in terms of r_P: *The representation π is (H, λ)-relatively cuspidal if and only if $r_P(\lambda) = 0$ for every proper σ-split parabolic subgroup P.*

5 Main theorem

Let (π, V) be a finitely generated H-distinguished admissible representation of G, with a non-zero H-invariant linear form $\lambda \in (V^*)^H$. Let $P = MU$ be a σ-split parabolic subgroup of G with the (σ, F)-split component S. We let $\mathcal{X}(S)$ denote the set of all quasi-characters of S and for a $\chi \in \mathcal{X}(S)$, we consider the *generalized χ-eigenspace* $(V_P)_{\chi,\infty}$ as in section 1.

Definition 5.1 A quasi-character $\chi \in \mathcal{X}(S)$ is called *an exponent of π_P relative to $r_P(\lambda)$* if the induced linear form $r_P(\lambda)$ on V_P is non-zero on the generalized χ-eigenspace $(V_P)_{\chi,\infty}$. The set of all such exponents is denoted by $\mathcal{E}_S(\pi_P; r_P(\lambda))$:

$$\mathcal{E}_S(\pi_P; r_P(\lambda)) = \left\{ \chi \in \mathcal{X}(S) \;\middle|\; r_P(\lambda)|_{(V_P)_{\chi,\infty}} \neq 0 \right\}.$$

Now we consider the following condition imposed on P and λ:

$(\natural_{P,\lambda})$ $\qquad |\chi(s)| < 1$ for all $\chi \in \mathcal{E}_S(\pi_P; r_P(\lambda))$ and all $s \in S^- \setminus Z_0\mathbf{S}(\mathcal{O}_F)$.

The main theorem of [KT2] is the following:

Theorem 5.2 ([KT2, 4.7]) *Let (π, V) be a finitely generated H-distinguished admissible representation of G, with a non-zero H-invariant linear form $\lambda \in (V^*)^H$. Then, the representation (π, V) is (H, λ)-square integrable if and only if the condition $(\natural_{P,\lambda})$ is satisfied for every σ-split parabolic subgroup P of G.*

Remark **5.3** By combining our criterion and Casselman's criterion, we have the following (possibly non-trivial) corollary: *If (π, V) is H-distinguished and is square integrable in the usual sense, then it is (H, λ)-square integrable for any $\lambda \in (V^*)^H$.*

6 Ingredients of the proof

To evaluate the L^2-norm of the (H, λ)-matrix coefficients, we first decompose G/Z_0H according to the analogue of Cartan decomposition. We fix a σ-stable open compact subgroup K_0 of G which has Iwahori factorization with respect to each σ-split parabolic subgroup. This is not a maximal compact subgroup. [BO] and [DS] gave the following: *There is a finite set Ξ of G and a finite set Γ of $(\mathbf{M_0 H})(F)$ such that*

$$G = \bigcup_{\xi \in \Xi, \gamma \in \Gamma} \bigcup_{s \in S_0^-} \xi K_0 s^{-1} \gamma H.$$

Choose ε so that Proposition 4.1 is valid for all I and put

$$G_{I,\gamma} = \bigcup_{s \in {}_I S_0^-(\varepsilon)/Z_0\mathbf{S}_0(\mathcal{O})} K_0 s^{-1}\gamma H$$

for each σ-split subset $I \subset \Delta$ and $\gamma \in \Gamma$. Then we have

$$G/Z_0H = \bigcup_{\xi,\gamma,I} \xi G_{I,\gamma}/Z_0H$$

by Lemma 3.3. Now, the evaluation of the L^2-norm of $\varphi = \varphi_{\lambda,v}$ starts from

$$\int_{G/Z_0H} |\varphi(g)|^2 dg \leqq \sum_{\xi,\gamma,I} \left(\int_{\xi G_{I,\gamma}/Z_0H} |\varphi(g)|^2 dg \right).$$

We may drop ξ by changing vector suitably. To prove the *if part* of the main theorem, it is enough to show the following:

Claim. *If the condition ($\sharp_{P,\lambda}$) is satisfied for $P = \gamma^{-1}P_I\gamma$ (see Lemma 3.2 (ii)), then*

$$\int_{G_{I,\gamma}/Z_0H} |\varphi(g)|^2 dg < \infty.$$

The integral is bounded by the series

$$\sum_{s \in {}_I S_0^-(\varepsilon)/Z_0\mathbf{S}_0(\mathcal{O})} \int_{K_0 s^{-1}\gamma Z_0H/Z_0H} |\varphi(g)|^2 dg$$

over the lattice ${}_I S_0^-(\varepsilon)/Z_0\mathbf{S}_0(\mathcal{O})$. Each term can be evaluated as

$$\int_{K_0 s^{-1}\gamma Z_0H/Z_0H} |\varphi(g)|^2 dg \leqq C \cdot \mathrm{vol}(K_0 s^{-1}\gamma Z_0H/Z_0H) \cdot |\varphi(s^{-1}\gamma)|^2$$

by a constant C. Look at the term where $\gamma = 1$ for simplicity. In the right hand side,

$$\varphi(s^{-1}) = \langle \lambda, \pi(s)v \rangle = \delta_P(s)^{1/2} \langle r_P(\lambda), \pi_P(s) j_P(v) \rangle$$

by Proposition 4.1. To proceed further, we need the following volume computation.

Lemma 6.1 ([KT2, 2.6]) *For each $\gamma \in \Gamma$, there exist positive real constants c_1 and c_2 such that*

$$c_1 \cdot \delta_{P_0}(s^{-1}) \leqq \mathrm{vol}(K_0 s^{-1}\gamma Z_0H/Z_0H) \leqq c_2 \cdot \delta_{P_0}(s^{-1})$$

for all $s \in S_0^-$.

From this lemma, the volume factor can be replaced by (a constant times) $\delta_{P_0}(s^{-1})$, so the problem reduces to the series

$$\sum_{s\in {}_IS_0^-(\varepsilon)/Z_0\mathbf{S}_0(\mathcal{O})} \left|\langle r_P(\lambda), \pi_P(s)j_P(v)\rangle\right|^2,$$

whose convergence will follow from the condition $(\sharp_{P,\lambda})$.

7 Examples

7.1 The symmetric space $GL_2(E)/GL_2(F)$ where E/F is quadratic

Let E/F be a quadratic extension of non-archimedean local fields and $\omega_{E/F}$ be the unique non-trivial character of $F^\times$ trivial on the norm image $N_{E/F}(E^\times)$.

Let G be the group $GL_2(E)$ and consider the involution σ on G defined by

$$\sigma(g) = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\overline{g}\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right),$$

where $\overline{(\cdot)}$ denotes the conjugation over F. Then the subgroup H of σ-fixed points in G is isomorphic to $GL_2(F)$. The standard parabolic subgroup $P = \{\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \in G\}$ is σ-split, with the σ-stable Levi subgroup $T = \{\left(\begin{smallmatrix} * & 0 \\ 0 & * \end{smallmatrix}\right) \in G\}$. Any proper σ-split parabolic subgroup is H-conjugate to P.

Non-cuspidal irreducible H-distinguished representations of G are completely determined in Hakim's thesis [H]:

1. The irreducible principal series $\mathrm{Ind}(\chi_1, \chi_2)$, with $\chi_2 = \overline{\chi_1}^{-1}$.
2. The irreducible principal series $\mathrm{Ind}(\chi_1, \chi_2)$, with $\chi_i|_{F^\times} \equiv 1$ and $\chi_1 \neq \chi_2$.
3. The spacial representation $\mathrm{sp}(\chi_1, \chi_2)$, with $\chi_1\chi_2^{-1} = |\cdot|_E$ and $\chi_1|\cdot|_E^{-1/2} = \chi_2|\cdot|_E^{1/2} = \omega_{E/F}$ on $F^\times$.

Here, for quasi-characters χ_1, χ_2 of $E^\times$, $\mathrm{Ind}(\chi_1, \chi_2)$ stands for the normalized induction determined by the quasi-character $\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right) \mapsto \chi_1(a)\chi_2(d)$ of T. In any case, it is known that the dimension of the space of H-invariant linear forms is one.

The representations in the second class are (H, λ)-relatively cuspidal (see Remark 2.2 for the definition, and Remark 4.2 for the criterion). Indeed, the Jacquet module along P is the direct sum $(\chi_1, \chi_2) \oplus (\chi_2, \chi_1)$ of characters of T. If $\chi_i|_F \equiv 1$, then we have

$$\chi_1 = \overline{\chi_1}^{-1}, \quad \chi_2 = \overline{\chi_2}^{-1},$$

so $\chi_2 \neq \overline{\chi_1}^{-1}$ provided that $\chi_1 \neq \chi_2$. Thus, neither (χ_1, χ_2) nor (χ_2, χ_1) cannot be trivial on $T \cap H = \{\left(\begin{smallmatrix} a & 0 \\ 0 & \bar{a} \end{smallmatrix}\right)\}$.

The representations in the third class provide examples of Remark 5.3.

7.2 The symmetric space $GL_3(F)/(GL_2(F) \times GL_1(F))$

Let G be the group $GL_3(F)$ and σ the inner involution $\sigma = \mathrm{Int}(\epsilon)$ defined by the anti-diagonal permutation matrix $\epsilon = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}$. Then the σ-fixed point subgroup H is isomorphic to $GL_2(F) \times GL_1(F)$. For this symmetric space, all the irreducible H-distinguished representations were determined by D. Prasad [P].

Form the normalized induction

$$\pi(\rho) = \mathrm{Ind}_{P_{1,2}}^{G}(1_{GL_1(F)} \otimes \rho)$$

from the standard parabolic $P_{1,2}$ of type $(1,2)$ and an irreducible representation ρ of $GL_2(F)$. Then $\pi(\rho)$ is irreducible, and is H-distinguished. The Borel subgroup P_0 consisting of upper triangular matrices is the only proper σ-split parabolic of G up to H-conjugacy. It is easy to determine exponents of $\pi(\rho)$ along P_0. By using our criterion, we may conclude that *$\pi(\rho)$ belongs to the discrete series for G/H if ρ is the Steinberg representation of $GL_2(F)$*. See [KT2, 5.1] for details.

7.3 The symmetric space $GL_4(F)/Sp_2(F)$

Let G be the group $GL_4(F)$ and σ the involution on G defined by

$$\sigma(g) = \begin{pmatrix} \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} & \\ & \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \end{pmatrix} {}^t g^{-1} \begin{pmatrix} \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} & \\ & \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \end{pmatrix}^{-1} \quad (g \in G).$$

Then the σ-fixed point subgroup H is the symplectic group $Sp_2(F)$. For this symmetric space, H-distinguished representations were studied by Heumos-Rallis [HR]. Let ρ be an irreducible admissible representation of $GL_2(F)$ and form the normalized induction

$$\mathrm{I}(\rho) = \mathrm{Ind}_{P_{2,2}}^{G}(\rho \cdot |\det(\cdot)|_F^{1/2} \otimes \rho \cdot |\det(\cdot)|_F^{-1/2}).$$

where $P_{2,2}$ is the standard parabolic of type $(2,2)$. Then $\mathrm{I}(\rho)$ has the unique irreducible quotient $\pi(\rho)$ which is H-distinguished ([HR, 11.1(b)]). One can show that *$\pi(\rho)$ belongs to the discrete series for G/H if ρ is the Steinberg representation of $GL_2(F)$*. See [KT2, 5.2] for details.

References

[BO] Y. Benoist and H. Oh, *Polar decomposition for p-adic symmetric spaces,* Int. Math. Res. Not. **2007(24)** (2007), 1–20.

[C] W. Casselman, *Introduction to the theory of admissible representations of p-adic reductive groups*, Unpublished manuscript, 1974, available at the website `http://www.math.ubc.ca/ people/faculty/cass/research.html`

[D] P. Delorme, *Constant term of smooth H_ψ-spherical functions on a reductive p-adic group*, Trans. Amer. Math. Soc. **362(2)** (2010), 933–955.

[DS] P. Delorme and V. Sécherre, *An analogue of the Cartan decomposition for p-adic reductive symmetric spaces,* Pacific J. Math. **251(1)** (2011), 1–21.

[H] J. Hakim, *Distinguished p-adic representations,* Duke Math. J. **62(1)** (1991), 1–22.

[HH] A.G. Helminck and G.F. Helminck, *A class of parabolic k-subgroups associated with symmetric k-varieties,* Trans. Amer. Math. Soc. **350** (1998), 4669–4691.

[HR] M. J. Heumos and S. Rallis, *Symplectic-Whittaker models for Gl_n,* Pacific J. Math. **146** (1990), 247–279.

[KT1] S. Kato and K. Takano, *Subrepresentation theorem for p-adic symmetric spaces*, Int. Math. Res. Notices **2008(12)** (2008), 1–40.

[KT2] S. Kato and K. Takano, *Square integrability of representations on p-adic symmetric spaces*, J. Funct. Anal. **258** (2010), 1427–1451.

[L] N. Lagier, *Terme constant de fonctions sur un espace symétrique réductif p-adique*, J. Funct. Anal. **254** (2008), 1088–1145.

[P] D. Prasad, *On the decomposition of a representation of $GL(3)$ restricted to $GL(2)$ over a p-adic field,* Duke Math. J. **69(1)** (1993), 167–177.